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Differential operators and immersions of a Riemann surface into a Grassmannian

Indranil Biswas

School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay 400005, India

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Abstract

We consider equivariant holomorphic immersions of a universal cover $\tilde{X}$ of a compact Riemann surface X into a Grassmannian $G(n, \mathbb{C}^{2n})$ satisfying a nondegeneracy condition. The equivariance condition says that there is a homomorphism ρ of the Galois group to $\mathrm{GL}(2n, \mathbb{C})$ that takes the natural action of the Galois group on $\tilde{X}$ to the action of the Galois group on $G(n, \mathbb{C}^{2n})$ defined using ρ. We prove that the space of such embeddings are in bijective correspondence with the space of all holomorphic differential operators of order two on a rank n vector bundle over X with the property that the symbol of the operator is an isomorphism.

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1. Introduction

Let X be a compact connected Riemann surface of genus g. Fix a universal cover $\tilde{X}$ of X. Let $\pi : \tilde{X} \to X$ be the projection map. The group of deck transformations will be denoted by Γ. So, $X = \tilde{X}/\Gamma$.

Let V denote a complex vector space of dimension $2n$. Let $G := G(n, V)$ be the Grassmannian of all n dimensional subspaces of V. The holomorphic tangent space to G at a point representing a subspace $F \subset V$ is $\mathrm{Hom}(F, V/F)$. Therefore, given a holomorphic map

$$f : \tilde{X} \to G, \tag{1.1}$$

the differential $\mathrm{d}f(x)$ at any point $x \in \tilde{X}$ gives a homomorphism

$$\widetilde{\mathrm{d}f}(x) : T_x\tilde{X} \otimes F \to V/F, \tag{1.2}$$

where $F \subset V$ is the subspace represented by $f(x)$.

E-mail address: indranil@math.tifr.res.in (I. Biswas).

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We will call the map f to be *nondegenerate* if $\widetilde{\mathrm{d}f}(x)$ is an isomorphism at every $x \in \tilde{X}$.

Let $\rho : \Gamma \to \mathrm{GL}(V)$ be a homomorphism. The map f in (1.1) is called *equivariant* with respect ρ if it commutes with the actions of Γ on $\tilde{X}$ and $G(n, V)$. Note that using ρ, the natural action of $\mathrm{GL}(V)$ on $G(n, V)$ induces an action of Γ on $G(n, V)$.

Take two such pairs $\mathfrak{t} := (f, \rho)$ and $\mathfrak{t}' := (f', \rho')$. So f (respectively, f') is equivariant with respect to ρ (respectively, ρ'). We will call $\mathfrak{t}$ to be *equivalent* to $\mathfrak{t}'$ if there is an automorphism $T \in \mathrm{GL}(V)$ that satisfies the following two conditions: $T \cdot f = f'$ and $T \cdot \rho \cdot T^{-1} = \rho'$.

Let $\mathcal{A}$ denote the space of all equivalence classes of all pairs (f, ρ), where ρ is a homomorphism and f is a nondegenerate map as in (1.1) equivariant with respect to ρ.

Let E and F be two holomorphic vector bundles over X and

$$D \in H^0(X, \mathrm{Diff}^2_X(E, F)) \tag{1.3}$$

be a global differential operator of order 2. The *symbol* $\sigma(D)$ of D is a homomorphism from $E \otimes K_X^{\otimes 2}$ to F, where K_X is the holomorphic cotangent bundle of X. The symbol map is defined in Section 2.

We assume that the operator D in (1.3) has the property that the symbol $\sigma(D)$ is an isomorphism. So, in particular $F \cong K_X^{\otimes 2} \otimes E$. Another such operator

$$D' \in H^0(X, \mathrm{Diff}^2_X(E', K_X^{\otimes 2} \otimes E'))$$

will be called equivalent to D if there is an isomorphism $T : E \to E'$ such that the following diagram commutes

$$\begin{array}{ccc} \underline{E} & \xrightarrow{D} & \underline{K_X^{\otimes 2} \otimes E} \\ \Big\downarrow T & & \Big\downarrow \mathrm{Id} \otimes T \\ \underline{E'} & \xrightarrow{D'} & \underline{K_X^{\otimes 2} \otimes E'} \end{array}$$

Here $\underline{W}$ denotes the sheaf of local holomorphic sections of a holomorphic vector bundle W.

Let $\mathcal{B}'$ denote the space of equivalence classes of such differential operators.

In Theorem 3.1, we construct a map

$$\mathcal{F} : \mathcal{B}' \to \mathcal{A},$$

which turns out to be surjective.

In Theorem 4.1, we construct an injective map

$$\mathcal{F}' : \mathcal{A} \to \mathcal{B}'.$$

The map $\mathcal{F}$ is the left inverse of $\mathcal{F}'$. In other words, the composition $\mathcal{F} \cdot \mathcal{F}'$ is the identity map of $\mathcal{A}$. The map $\mathcal{F}$ is not injective if $g \geq 1$.

Let L be a holomorphic line bundle over X equipped with a flat connection. Given any $D \in \mathcal{B}'$ as in (1.3), using the flat connection on L, the operator D gives another operator

$$D' \in H^0(X, \mathrm{Diff}^2_X(E \otimes L, K_X^{\otimes 2} \otimes E \otimes L)) \in \mathcal{B}'.$$

Let $\mathcal{B}$ denote the space of all equivalence classes of elements in $\mathcal{B}'$, where D is equivalent to D' if there is a flat line bundle L such that D' is obtained from D in the above fashion.

The map $\mathcal{F}$ (or $\mathcal{F}'$) induces a bijective correspondence between the two spaces $\mathcal{A}$ and $\mathcal{B}$ (Theorem 4.2).

Many interesting results on maps of a curve into a Grassmannian can be found in [2]. In fact, reading [2] inspired to look into maps of curves to Grassmannians. In [4] we prove similar results for embeddings in the Grassmannian of r-dimensional subspaces in $\mathcal{C}^{nr}$.

2. Construction of connection from differential operator

We briefly recall the definition of jet bundles and its basic properties.

Let E be a holomorphic vector bundle over X, and let k be a nonnegative integer. The kth order *jet bundle* of E, denoted by $J^k(E)$, is defined to be the following direct image on X:

$$J^k(E) := p_1^* \left(\frac{p_2^* E}{p_2^* E \otimes \mathcal{O}_{X\times X}(-(k+1)\Delta)} \right),$$

where $p_i : X \times X \to X$, $i = 1, 2$, is the projection onto the ith factor and Δ is the diagonal divisor on $X \times X$ consisting of all points of the form (x, x). There is a natural exact sequence

$$0 \to K_X^{\otimes k} \otimes E \to J^k(E) \to J^{k-1}(E) \to 0 \tag{2.1}$$

which is constructed using the obvious inclusion of $\mathcal{O}_{X\times X}(-(k+1)\Delta)$ in $\mathcal{O}_{X\times X}(-k\Delta)$. The inclusion map $K_X^{\otimes k} \otimes E \to J^k(E)$ is constructed by using the homomorphism

$$K_X^{\otimes k} \to J^k(\mathcal{O}_X),$$

which is defined at any $x \in X$ by sending $(\mathrm{d}f)^{\otimes k}$, where f is any holomorphic function with $f(x) = 0$, to the jet of the function $f^k/k!$ at x. Any homomorphism $E \to F$ induces a homomorphism

$$J^k(E) \to J^k(F) \tag{2.2}$$

for any $k \geq 0$.

The sheaf of *differential operators* $\mathrm{Diff}_X^k(E, F)$ is defined to be $\mathrm{Hom}(J^k(E), F)$. The homomorphism

$$\sigma : \mathrm{Diff}_X^k(E, F) \to \mathrm{Hom}(K_X^{\otimes k} \otimes E, F)$$

obtained by restricting a homomorphism from $J^k(E)$ to F to the subsheaf $K_X^{\otimes k} \otimes E$ in (2.1) is known as the *symbol map*.

So, for any $D \in H^0(X, \mathrm{Diff}_X^2(E, F))$ the symbol $\sigma(D)$ is a homomorphism from $K_X^{\otimes 2} \otimes E$ to F. This proves the assertion in the introduction that $F \cong K_X^{\otimes 2} \otimes E$ for any $D \in \mathcal{B}'$. In that case, using this isomorphism, the symbol of D is the identity automorphism of E.

For all $k, l \geq 0$, there is a natural injective homomorphism

$$\tau : J^{k+l}(E) \to J^k(J^l(E)). \tag{2.3}$$

We will describe the image of τ for the special case $k = 1 = l$. Using (2.2), the homomorphism $J^1(E) \to E$ in (2.1) gives a homomorphism $\gamma : J^1(J^1(E)) \to J^1(E)$. On the other hand, (2.1) gives a homomorphism $\gamma' : J^1(J^1(E)) \to J^1(E)$. The image $\tau(J^2(E))$ is the kernel of the difference $\gamma - \gamma'$. In other words, we have an exact sequence

$$0 \to J^2(E) \xrightarrow{\tau} J^1(J^1(E)) \xrightarrow{\gamma-\gamma'} K_X \otimes E \to 0. \tag{2.4}$$

Note that the image of $\gamma - \gamma'$ is contained in the subbundle $K_X \otimes E \subset J^1(E)$ since the two projections of $J^1(J^1(E))$ to E, obtained from γ and γ', respectively, coincide.

Let $D \in H^0(X, \mathrm{Diff}^2_X(E, E \otimes K_X^{\otimes 2}))$ as in (1.3) be an operator in $\mathcal{B}'$. Consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K^{\otimes 2} \otimes E & \longrightarrow & J^2(E) & \longrightarrow & J^1(E) & \longrightarrow & 0 \\ & & \big\downarrow & & \big\downarrow \tau & & \| & & \\ 0 & \longrightarrow & K_X \otimes J^1(E) & \longrightarrow & J^1(J^1(E)) & \longrightarrow & J^1(E) & \longrightarrow & 0 \end{array} \tag{2.5}$$

where τ is defined in (2.3).

Now, since the symbol of D is the identity automorphism of E, the operator D, which is a homomorphism from $J^2(E)$ to $K^{\otimes 2} \otimes E$, gives a splitting of the top exact sequence in (2.5). Let

$$f_D : J^1(E) \to J^2(E)$$

be the homomorphism given by this splitting. The composition $\tau \cdot f_D$ is splitting of the bottom exact sequence in (2.5).

A splitting of the bottom exact sequence in (2.5) is a holomorphic connection on E [1]. Since $\dim_{\mathbb{C}} X = 1$, any holomorphic connection on X is flat. Therefore, the operator D gives a flat connection on $J^1(E)$.

Let ∇^D denote the flat connection on $J^1(E)$ obtained from D.

Consider the subbundle $K_X \otimes E$ of $J^1(E)$ given by (2.1). Its second fundamental form for the connection ∇^D gives a homomorphism

$$\phi : E \to E.$$

If s is a local holomorphic section of E defined around a point $x \in X$, then ϕ sends $s(x)$ to the projection on E_x of $\nabla^D_v(v^* \otimes s) \in J^1(E)_x$, where $v \in T_x X$ is any nonzero tangent vector and v^* is any local holomorphic section of K_X such that $v^*(x)$ is the dual of v.

Proposition 2.1. *The second fundamental form ϕ is the identity automorphism of E.*

Proof. The proof involves a tedious unraveling of the various definitions.

Take a point $x \in X$ and take a vector $v \in (K_X \otimes E)_x$ in the fiber over x. Consider

$$f_D(v) \in J^2(E)_x,$$

where f_D defined above is the splitting homomorphism for the top exact sequence in (2.5). Let s be a local holomorphic section of E defined around x such that the element in the fiber

$J^2(E)_x$ representing s coincides with $f_D(v)$. The commutativity of the right-hand square in (2.5) implies that $s(x) = 0$ and the element in $J^1(E)_x$ representing s is v.

Consider the commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & K_X^{\otimes 2} \otimes E \arrow[r] \arrow[d] & J^1(K_X \otimes E) \arrow[r] \arrow[d] & K_X \otimes E \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K_X \otimes J^1(E) \arrow[r] \arrow[d] & J^1(J^1(E)) \arrow[r] \arrow[d, "\gamma"] & J^1(E) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K_X \otimes E \arrow[r, "\delta"] \arrow[d] & J^1(E) \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

where all the exact sequences (horizontal or vertical), except the middle vertical one, are obtained from (2.1), and all the homomorphisms in the middle vertical exact sequence are obtained from (2.2). The above homomorphism γ is the one constructed in (2.1). Recall that γ is the homomorphism in (2.2) for the projection $J^1(E) \to E$.

Take a local holomorphic section u of the subbundle $K_X \otimes E$ of $J^1(E)$ such that $u(x) = v$. The section of $J^1(J^1(E))$ representing u will be denoted by $\overline{u}$. From the exactness of the middle vertical sequence it follows that $\gamma(\overline{u}) = 0$.

Recall that $J^2(E)_x$ is a subspace of $J^1(J^1(E))_x$. The image $\gamma(f_D(v))$ clearly coincides with the image $\delta(v)$, where δ is defined in the above diagram. In view of the earlier remark that $\gamma(\overline{u}) = 0$, from the definition of ϕ it follows immediately that if $v = \omega \otimes e$, where $\omega \in (K_X)_x$ and $e \in E$, then $\phi(e) = e$. This completes the proof of the proposition. □

In Section 3 using the connection ∇^D, we will construct a nondegenerate immersion of the universal cover $\tilde{X}$ in $G(n, V)$.

3. Relationship between connections and immersions

We continue with the notation set up in Sections 1 and 2.

Consider $\pi^* J^1(E)$ on $\tilde{X}$. Fix a point $y \in \tilde{X}$ together with an isomorphism of $\pi^* J^1(E)_y$ with the vector space V. Using the connection $\pi^*\nabla^D$, the vector bundle $\pi^* J^1(E)$ gets identified with the trivial vector bundle over $\tilde{X}$ with fiber V.

The monodromy of ∇^D gives a homomorphism

$$\rho : \Gamma = \pi_1(X, \pi(y)) \to \mathrm{Aut}(E_{\pi(y)}) = \mathrm{GL}(V) \tag{3.1}$$

A different choice of the isomorphism between $\pi^* J^1(E)_y$ and V sends ρ to the composition of ρ with an inner conjugation of $\mathrm{GL}(V)$.

Let

$$f : \tilde{X} \to G := G(n, V) \tag{3.2}$$

be the holomorphic map that sends any $z \in \tilde{X}$ to the subspace

$$\pi^*(K_X \otimes E)_z \subset \pi^* J^1(E)_z = V.$$

From its definition, it is immediate that the map f is equivariant with respect to ρ defined in (3.1).

Our next goal is to show that f is nondegenerate in the sense defined in the introduction.

Proposition 3.1. *The homomorphism* $\widetilde{\mathrm{d}f}$ *defined in* (1.2) *is an isomorphism. In other words, the map f is nondegenerate.*

Proof. First observe that the homomorphism $\widetilde{\mathrm{d}f}$ coincides with the second fundamental form for the subbundle $\pi^*(K_X \otimes E)$ of the flat vector bundle $\pi^* J^1(E)$. This is obvious. Indeed, if

$$0 \to S \to \overline{V} \to Q \to 0$$

is the universal exact sequence on $G(n, V)$, where $\overline{V}$ is the trivial vector bundle over $G(n, V)$ with fiber V, then the second fundamental form of S for the natural flat connection on $\overline{V}$ is the canonical identification $\mathrm{T}G(n, V) \cong \mathrm{Hom}(S, Q)$.

The second fundamental form of a subbundle is compatible with the pullback operation. In other words, $\widetilde{\mathrm{d}f}$ coincides with $\pi^*\phi$. Therefore, from Proposition 2.1, it follows that $\widetilde{\mathrm{d}f}$ is an isomorphism. This completes the proof of the proposition. □

Note that from the proof of Proposition 3.1 it follows immediately that if $\widetilde{\mathrm{d}f}$ is an isomorphism then the second fundamental form for the restriction to $\tilde{X}$ of the above universal exact sequence over $G(n, V)$ is also an isomorphism.

The element $(f, \rho) \in \mathcal{A}$ constructed in (3.1) and (3.2) does not depend on the choice of the point y or on the choice of the isomorphism between $\pi^* J^1(E)_y$ and V.

Therefore, summarizing the construction we have

Theorem 3.1. *There is a canonical map*

$$\mathcal{F} : \mathcal{B}' \to \mathcal{A}$$

that sends a differential operator D to the pair (f, ρ) *constructed above from D.*

We will now derive a consequence of the isomorphism condition of second fundamental forms.

Let W be a holomorphic vector bundle of rank $2n$ over X equipped with a flat connection ∇. Let

$$0 \to S' \xrightarrow{i} W \xrightarrow{q} Q' \to 0 \tag{3.3}$$

be an exact sequence of holomorphic vector bundles with $\mathrm{rank}(S') = n = \mathrm{rank}(Q')$. There is a natural homomorphism

$$\beta : W \to J^1(Q') \tag{3.4}$$

which is defined as follows. For any $x \in X$ and $v \in W_x$, let s denote the (unique) local flat section of W defined around x with $s(x) = v$. The map β sends v to the element in $J^1(Q')_x$ representing $q \cdot s$, where q is the projection in (3.3).

Let

$$\psi : S' \to K_X \otimes Q',$$

be the second fundamental form of the subbundle S' for ∇.

Lemma 3.1. *If the second fundamental form ψ is an isomorphism, then the homomorphism β defined in* (3.4) *is also an isomorphism.*

Proof. The following diagram evidently commutes

$$\begin{array}{ccc} W & \xrightarrow{\beta} & J^1(Q') \\ \big\downarrow q & & \big\downarrow \\ Q' & = & Q' \end{array}$$

Let $\overline{\beta} : S' \to K_X \otimes Q'$ be the homomorphism obtained by restricting β to S'.

Comparing the definitions of β and ψ it is easy to see that $\overline{\beta}$ coincides with ψ. This completes the proof the lemma. $\square$

Let E be a stable vector bundle of rank n and degree $n(1-g)$. See [3] about stable bundles.

Since E is stable, we have $H^0(X, \mathrm{End}(E)) = \mathbb{C}$, and hence $H^1(X, K_X \otimes \mathrm{End}(E)) = \mathbb{C}$. Let

$$0 \to K_X \otimes E \xrightarrow{i} W \xrightarrow{q} E \to 0 \tag{3.5}$$

be the unique nontrivial extension.

Lemma 3.2. *The extension W admits flat connections. For every flat connection ∇ on W, the second fundamental form of $K_X \otimes E$ is an isomorphism.*

Proof. A result of A. Weil says that a holomorphic vector bundle W_0 over X admits a flat connection if and only if every direct summand of W_0 is of degree zero. Therefore, to prove that W admits a flat connection it suffices to show that W is not a direct sum of vector bundles.

Assume that $W = W_1 \oplus W_2$. Take two nonzero distinct scalars μ_1 and μ_2, and let T denote the automorphism of W that acts as multiplication by μ_i on W_i.

Consider the composition $q \cdot T \cdot i : K_X \otimes E \to E$, where q and i are as in (3.5). Since E is stable and degree $(K_X \otimes E) >$ degree(E), we have $q \cdot T \cdot i = 0$. Now, since E is simple, $K_X \otimes E$ must be contained in either W_1 or W_2.

Since $q \cdot T \cdot i = 0$, the automorphism T induces an automorphism of the quotient E in (3.5). Since E is stable, the induced automorphism of E must be a scalar multiplication. This implies that $K_X \otimes E$ must coincide with either W_1 or W_2. If $K_X \otimes E = W_1$, then the eigenspace of W for the eigenvalue μ_2 gives a splitting of (3.5). This contradicts the nontriviality of the extension and hence W admits a flat connection.

Let ∇ be a flat connection on W. Now, since degree$(K_X \otimes E) \neq 0$, it does not admit any flat connection. Therefore, the connection ∇ does not preserve $K_X \otimes E$. In other words, the second fundamental form of $K_X \otimes E$ must be nonzero. Finally, since the second fundamental form is an endomorphism of E, and E is simple, the endomorphism must be an isomorphism. This completes the proof of the lemma. □

We note that now Lemma 3.1 says that $W \cong J^1(E)$.

A polystable vector bundle is a direct sum of stable vector bundles of same quotient degree/rank. If E is polystable of degree $n(1-g)$, then clearly Lemma 3.2 remains valid for E. Indeed, a connection on a direct sum of vector bundles induces a connection on each direct summand.

In the first part of this section, we saw that a flat connection on $J^1(E)$, with the second fundamental form an isomorphism, gives an element $(f, \rho) \in \mathcal{A}$. Therefore, from the space of all pairs (W, ∇), where W is polystable and ∇ is a flat connection on $J^1(W)$, we have a map to $\mathcal{A}$.

If genus$(X) \geq 2$, then degree$(K_X) > 0$. Consequently, for any polystable vector bundle E over X, the sequence (3.5) defines the Harder–Narasimhan filtration of $W = J^1(E)$. Therefore, for another polystable vector bundle E', if $J^1(E')$ is isomorphic to $J^1(E)$, then E must be isomorphic to E'.

In Section 4 we will construct a map from $\mathcal{A}$ to $\mathcal{B}'$.

4. Construction of differential operators from immersions

Let $(f, \rho) \in \mathcal{A}$. Consider the pullback

$$0 \to f^*S \to f^*\overline{V} \to f^*Q \to 0$$

on $\tilde{X}$ of the universal exact sequence of $G(n, V)$. Since this exact sequence is equivariant for the action of Γ through ρ, it descends to X as an exact sequence

$$0 \xrightarrow{i} F \to W \xrightarrow{q} E \to 0 \tag{4.1}$$

The flat connection on $f^*\overline{V}$ descends to a flat connection ∇ on W.

We noted in the previous section that the nondegeneracy condition of the map f ensures that the second fundamental form for the subbundle f^*S of the flat bundle $f^*\overline{V}$ is an isomorphism. Therefore, the second fundamental form of the subbundle F in (4.1) for ∇ is an isomorphism.

For any point $x \in X$ and any vector $v \in W_x$ in the fiber, let $\overline{v}$ denote the unique flat local section of W, defined around x, such that $\overline{v}(x) = v$. For any integer $k \geq 1$, let

$$\theta_k : W \to J^k(E) \tag{4.2}$$

denote the homomorphism that sends any v to the element in $J^k(E)_x$ representing the local section $q(\overline{v})$ of E, where q is defined in (4.1). It is easy to see that the following diagram is commutative

$$\begin{array}{ccc} W & \xrightarrow{\theta_{k+1}} & J^{k+1}(E) \\ \| & & \downarrow \\ W & \xrightarrow{\theta_k} & J^k(E) \end{array} \tag{4.3}$$

where the homomorphism $J^{k+1}(E) \to J^k(E)$ is given by (2.1) and θ_k is defined in (4.2).

We already noted that the second fundamental form for F is an isomorphism. Consequently, Lemma 3.1 says that θ_1 is an isomorphism. Now consider the composition

$$\theta := \theta_2 \cdot \theta_1^{-1} : J^1(E) \to J^2(E).$$

The commutativity of (4.3) immediately implies that θ gives a splitting of the top exact sequence in (2.5).

Consequently, we have differential operator $D \in H^0(X, \mathrm{Diff}^2_X(E, K_X^{\otimes 2} \otimes E))$ defined by the homomorphism $J^2(E) \to K_X^{\otimes 2} \otimes E$ obtained from the splitting defined by θ. Since D is defined by a splitting of the jet sequence, its symbol is the identity automorphism of E.

It is easy to see that the element $\mathcal{F}(D) \in \mathcal{A}$, where $\mathcal{F}$ is constructed in Theorem 3.1, is equivalent to the pair (f, ρ) we started with.

Therefore, we now have

Theorem 4.1. *There is a canonical map*

$$\mathcal{F}' : \mathcal{A} \to \mathcal{B}'$$

that sends any pair (f, ρ) to the differential operator D constructed above. Furthermore, the composition $\mathcal{F} \cdot \mathcal{F}'$ is the identity map of $\mathcal{A}$. In particular, $\mathcal{F}'$ is injective and $\mathcal{F}$ is surjective.

It now remains to examine how $\mathcal{F}$ fails to be injective.

Let L be a holomorphic line bundle over X equipped with a flat connection ∇^L.

Take a differential operator $D \in H^0(X, \mathrm{Diff}^2_X(E, K_X^{\otimes 2} \otimes E))$ with symbol Id_E. Set $E' := E \otimes L$. Any local holomorphic section s' of E' can be expressed as $s \otimes l$, where s is a local holomorphic section of E and l is a *flat* local section of L. Construct a differential operator

$$D' \in H^0(X, \mathrm{Diff}^2_X(E', K_X^{\otimes 2} \otimes E'))$$

by sending every such local section s' of E' to the local section $D(s) \otimes l$ of $K_X^{\otimes 2} \otimes E'$. Since any two flat local sections of L differ by multiplication with a constant scalar, the differential operator D' is well-defined. It is also obvious that the symbol of D' is the identity automorphism of E'. Consequently, D' defines an element of $\mathcal{B}'$.

We will call two operators $D, D' \in \mathcal{B}'$ to be *equivalent* if there is some flat line bundle L such that D' is constructed from D in the above fashion. Let $\mathcal{B}$ denote the space of all equivalence classes.

From the construction of the map $\mathcal{F}$ in Theorem 3.1, it is immediate that it factors through the projection $\eta : \mathcal{B}' \to \mathcal{B}$. Let $\overline{\mathcal{F}} : \mathcal{B} \to \mathcal{A}$ be the map induced by $\mathcal{F}$. It is straight-forward to check that the two maps $\overline{\mathcal{F}}$ and $\eta \cdot \mathcal{F}'$ are inverses of each other.

Therefore, we now have

Theorem 4.2. *The two maps $\overline{\mathcal{F}}$ and $\eta \cdot \mathcal{F}'$ induce bijective correspondence between the two spaces $\mathcal{A}$ and $\mathcal{B}$, and they are inverses of each other.*

In Section 3, we saw that any pair (E, ∇), where E is a polystable vector bundle of rank n over X of degree $n(1-g)$ and ∇ is a flat connection on $J^1(E)$, gives an element of $\mathcal{A}$, and hence it gives an element of $\mathcal{B}$. It may be interesting to be able to characterize the subset of $\mathcal{A}$ (or $\mathcal{B}$) defined by such pairs. It is not clear whether it is a proper subset.

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